$$x^2 + y^2 = (a_1a_2a_3\cdots a_m)^n,$$

has, in Legendre's notation, $E((n+1)^m/2)$ solutions. Show that in $2^{m-1}$ of these solutions $x$ and $y$ are relatively prime.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

**445. Proposed by S. A. JOFFE, New York City.**

Sum the series

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a\binom{cn-ca}{a}.$$

### I. SOLUTION BY THE PROPOSER.

This is a generalization of the series forming the first member of equation (2) in the Proposer's solution of problem No. 424 (*June* 1915 issue, p. 205), obtained by multiplying the upper indices $n, n-1, n-2, \cdots$ by the constant factor $c$. Following the method employed in that solution, we find that the given series equals

$$\Delta^a\binom{cn-ca}{a},$$

the finite differences being taken with respect to $n$.

Now

$$\Delta_x\binom{cx}{a} = \binom{c\overline{x+1}}{a} - \binom{cx}{a},$$

the second member of which may be written in the following form:

$$\binom{cx+c}{a} - \binom{cx+c-1}{a} + \binom{cx+c-1}{a} - \binom{cx+c-2}{a} + \binom{cx+c-2}{a} - \cdots$$
$$+ \binom{cx+1}{a} - \binom{cx}{a},$$

since all these terms, except the first and last, alternately cancel each other. Combining the terms in pairs and noticing that

$$\binom{cx+c}{a} - \binom{cx+c-1}{a} = \binom{cx+c-1}{a-1},\quad \binom{cx+c-1}{a} - \binom{cx+c-2}{a} = \binom{cx+c-2}{a-1},$$

etc., we have

$$\Delta_x\binom{cx}{a} = \binom{cx+c-1}{a-1} + \binom{cx+c-2}{a-1} + \cdots + \binom{cx+1}{a-1} + \binom{cx}{a-1},$$

which means that the first difference, taken with respect to $x$, of the binomial coefficient $\binom{cx}{a}$ having for its lower index $a$, equals the sum of $c$ binomial coefficients, each having for its lower index $a-1$.

In the same manner, the second difference $\Delta_x{}^2\binom{cx}{a}$ may be expressed as the sum of $c\cdot c = c^2$ binomial coefficients, each having $a-2$ for its lower index; and continuing this process, we find that the $a$th difference $\Delta_x{}^a\binom{cx}{a}$ may be expressed as the sum of $c^a$ binomial coefficients, each having for its lower index $a-a=0$ and hence each equal to 1. In other words,

$$\Delta_x{}^a\binom{cx}{a} = c^a,$$

and, similarly,

$$\Delta_n{}^a\binom{c\,\overline{n-a}}{a} = c^a.$$

We thus arrive at the result:

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a\binom{cn-ca}{a} = c^a.$$

## II. Solution by Norman Anning, Chilliwack, B. C.

Define operators $E$ and $\Delta$ as follows:

$$E^h f(n) = f(n+h), \qquad \Delta f(n) = f(n+1) - f(n) = (E-1)f(n).$$

Then the given expression,

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a\binom{cn-ca}{a}$$

$$= (1-E^{-1})^a\binom{cn}{a} = (\Delta E^{-1})^a\binom{cn}{a} = \Delta^a E^{-a}\binom{cn}{a} = \Delta^a\binom{cn-ca}{a}.$$

Now

$$\binom{cn-ca}{a} = \frac{1}{a!}\{c^a n^a + \text{terms in } n \text{ of degree lower than } a\}.$$

Hence,

$$\Delta^a\binom{cn-ca}{a} = \frac{1}{a!}\{c^a \cdot a! + 0\} = c^a.$$

**446. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.**

Solve the equations

$$x^2(y-z) = l^2(m-n), \qquad y^2(z-x) = m^2(n-l), \qquad z^2(x-y) = n^2(l-m).$$

## Solution by H. S. Uhler, Yale University.

An equation involving only one variable $x$ may be obtained in the following manner. Equating the expressions for $z$ derived from the first two given equations we find

$$y - l^2(m-n)/x^2 = x + m^2(n-l)/y^2$$

or

$$x^3y^2 - x^2y^3 + m^2(n-l)x^2 + l^2(m-n)y^2 = 0. \tag{1}$$

Substituting the value of $z$ from the first in the third given equation and reducing, we obtain

$$x^5y^2 - x^4y^3 - n^2(l-m)x^4 - 2l^2(m-n)x^3y + 2l^2(m-n)x^2y^2 + l^4(m-n)^2x - l^4(m-n)^2y = 0. \tag{2}$$

Multiplying (1) by $x^2$ and subtracting (2) from the product gives, after removing the factor $m-n$,

$$(lm - mn + nl)x^4 - 2l^2x^3y + l^2x^2y^2 + l^4(m-n)x - l^4(m-n)y = 0. \tag{3}$$

Multiplying (1) by $l^2$, (3) by $y$, adding, and dividing by $x$,

$$(lm - mn + nl)x^3y - l^2x^2y^2 + l^2m^2(n-l)x + l^4(m-n)y = 0. \tag{4}$$

Adding (3) and (4) and dividing by $x$,

$$(lm - mn + nl)x^3 + (lm - mn + nl - 2l^2)x^2y + l^2[l^2(m-n) + m^2(n-l)] = 0. \tag{5}$$

Substituting the expression for $y$ from (5) in (4) and removing the factors $(l-m)(l-n)$ we derive the following quadratic in $x^3$, namely

$$(lm - mn + nl)^2x^6 - 2l^4[m^2(l-n) + n^2(l-m)]x^3 - l^6(mn - nl + lm)(nl - lm + mn) = 0.$$

Consequently

$$x^3 = l^3$$

and

$$x^3 = \frac{l^3[l^2(m-n)^2 - m^2n^2]}{(lm - mn + nl)^2}.$$